World Journal of Advanced Engineering Technology and Sciences
eISSN: 2582-8266
Cross Ref DOI: 10.30574/wjaets
Journal homepage: https://wjaets.com/

(RESEARCH ARTICLE)

Check for updates

Upper bounds for the multiplicative Y-index and S-index of some operations on graphs

Nagarajan Sethumadhavan and Kayalvizhi Gokulathilagan *

Department of Mathematics, Kongu Arts and Science College (Autonomous), Erode, Tamil Nadu, India.

World Journal of Advanced Engineering Technology and Sciences, 2024, 11(02), 245–256

Publication history: Received on 13 February 2024; revised on 21 March 2024; accepted on 23 March 2024

Article DOI: https://doi.org/10.30574/wjaets.2024.11.2.0100

Abstract

Topological index is a numerical descriptor of a molecule; it is found that there is strong correlation between the proerties of chemical compounds and their molecular structure based on a specific topological feature of the corresponding molecular graph. In this paper, we introduce two new graph invariants known as the Multiplicative Y - index and Multiplicative S-index of a graph. We establish the upper bounds for the Multiplicative Y-index and Multiplicative S-index of the graph operations such as Join, Cartesian product, Composition, Tensor product, Strong product, Disjunction, Symmetric difference, Corona product, Corona join product and the indices are evaluated for some well-known graphs.

Keywords: ***Y***-index, ***S***-index; Graph operations; Zagreb; Degree**.**

2020 Mathematics Subject Classification: 05C07, 05C76.

1. Introduction

Graph theory has given chemists many useful tools, such as topological indices. Molecules and molecular compounds are frequently represented by molecular graphs. A topological index can be thought of as the conversion of a chemical structure into a real number. Topological and graph invariants based on distances between graph vertices are widely used for characterizing molecular graphs, establishing relationships between structural and property, properties of molecules, predicting biological activities of chemical compounds, and developing chemical applications. Topological indices have the significance of being able to be used directly as simple numerical descriptors in comparison with physical, chemical, or biological parameters of molecules in Quantitative Structure Property Relationships (QSPR) and Quantitative Structure Activity Relationships (QSAR). There are several types of topological indices, including distance-based topological indices, degreebased topological indices, and counting-related polynomials and graph indices. In medicinal chemistry and bioinformatics, the current trend of numerical coding of chemical structures with topological indices or topological coindices has been quite successful.

Let's consider two simple connected graphs, G_m and G_n, each with disjoint vertex and edge sets. For $i = m, n$, g_i and h_i represent the number of vertices and edges. The degree of a vertex v is the number of edges incident on the vertex v and is expressed as $d_G(v) = \beta_G(v)$ for every vertex $v \in V(G)$.

In 1972, I. Gutman and N. Trinajstic [5] defined the first and second Zagreb index of a graph as:

* Corresponding author: Kayalvizhi Gokulathilagan.

$$M_1(G) = \sum_{v \in V(G)} [\beta_G(v)^2] = \sum_{uv \in E(G)} [\beta_G(u) + \beta_G(v)]$$

$$M_2(G) = \sum_{uv \in E(G)} [\beta_G(u)\beta_G(v)]$$

B. Furtula and I. Gutman defined the F-index as [4] in 2015:

$$F(G) = \sum_{v \in V(G)} [\beta_G(v)^3] = \sum_{uv \in E(G)} [\beta_G(u)^2 + \beta_G(v)^2]$$

In 2020, Abdu Alameri and Noman AI-Naggar [1] introduced the Y-index, which is defined as:

$$Y(G) = \sum_{v \in V(G)} [\beta_G(v)^4] = \sum_{uv \in E(G)} [\beta_G(u)^3 + \beta_G(v)^3]$$

In 2021, S. Nagarajan , G. Kayalvizhi and G. Priyadharsini defined the S-index as [8]:

$$S(G) = \sum_{v \in V(G)} [\beta_G(v)^5] = \sum_{uv \in E(G)} [\beta_G(u)^4 + \beta_G(v)^4]$$

In 2010, R. Todeschini and D. Ballabio [11] introduced the first and second Multiplicative Zagreb indices of a graph, which is defined as:

$\prod_1(G) = \prod_{v \in V(G)} \beta_G(v)^2$ and $\prod_2(G) = \prod_{uv \in E(G)} \beta_G(u)\beta_G(v)$

In 2019, Asghar Yousefi and Ali Iranmanesh [2] introduced the Multiplicative forgotten topological index, which is defined as:

$\prod_F(G) = \prod_{v \in V(G)} \beta_G(v)^3$

In (2013) C.D. Kinkar and Y. Aysum [7] derived graph operations in Multiplicative Zagreb indices . K. Xu and K.C. Das [12] computed the Multiplicative Zagreb coindices in (2013). In [2] Y. Asghar and Ali Iranmanesh (2019) derived the Multiplicative F-index of graph operations. In this paper, we evaluated few well-known graphs and expressions for the upper bounds for the Multiplicative Y -index and Multiplicative S-index of various graph operations. Investigators interested in learning more about graph operations can consult to [1,4,6,9,3,10,13].

Definition 1.1

The Multiplicative Y-index of a graph G is defined as the product of a graph's four degree vertices and is denoted by:

$\prod_Y(G) = \prod_{v \in V(G)} \beta_G(v)^4$

Definition 1.2

The Multiplicative S-index of a graph G is defined as the product of a graph's five degree vertices and is denoted by:

$\prod_S(G) = \prod_{v \in V(G)} \beta_G(v)^5$

Main Results

Lemma 2.1: [7] (AM-GM inequality)

Let $x_1, \ldots, x_n$ be a nonnegative numbers. Then

$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[k]{x_1, \ldots, x_n}$ holds with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Corollary 2.2: For a graph G with n vertices, we've

$$\prod_{Y}(\mathrm{P_n}) = 16^{\mathrm{n}-2},\ \ \mathrm{n} \geq 3$$

$$\prod_{Y}(\mathrm{C_n}) = 16^{\mathrm{n}},\ \ \mathrm{n} \geq 3$$

$$\prod_{Y}(\mathrm{S_n}) = (\mathrm{n}-1)^4,\ \ \mathrm{n} \geq 3$$

$$\prod_{Y}(\mathrm{W_n}) = 3^{4(\mathrm{n}-1)}(n-1)^4,\ \ \mathrm{n} \geq 3$$

$$\prod_{Y}(\mathrm{L_n}) = 3^{4(\mathrm{n}-1)}16^4,\ \ \mathrm{n} \geq 3$$

$$\prod_{Y}(\mathrm{K_n}) = (\mathrm{n}-1)^4 n,\ \ \mathrm{n} \geq 3$$

Corollary 2.3: For a graph G with n vertices, we've

$$\prod_{S}(\mathrm{P_n}) = 32^{\mathrm{n}-2},\ \ \mathrm{n} \geq 3$$

$$\prod_{S}(\mathrm{C_n}) = 32^{\mathrm{n}},\ \ \mathrm{n} \geq 3$$

$$\prod_{S}(\mathrm{S_n}) = (\mathrm{n}-1)^5,\ \ \mathrm{n} \geq 3$$

$$\prod_{S}(\mathrm{W_n}) = 3^{5(\mathrm{n}-1)}(n-1)^5,\ \ \mathrm{n} \geq 3$$

$$\prod_{S}(\mathrm{L_n}) = 3^{5(\mathrm{n}-1)}32^4,\ \ \mathrm{n} \geq 3$$

$$\prod_{S}(\mathrm{K_n}) = (\mathrm{n}-1)^5 n,\ \ \mathrm{n} \geq 3$$

1.1. The Join of graph

The join $G_m + G_n$ of graphs G_m and G_n with vertex sets $V(G_m)$ and $V(G_n)$ and edge sets $E(G_m)$ and $E(G_n)$ is the graph union $G_m \cup G_n$ together with all the edges between $V(G_m)$ and $V(G_n)$. Obviously$|V(G_m + G_n)| = |V(G_m)| + |V(G_n)| = p_m + p_n$,$|E(G_m + G_n)| = |E(G_m)| + |E(G_n)| + |V(G_m)||V(G_n)| = q_m + q_n + p_m p_n$.

$$\beta_{G_m+G_n}(v) = \begin{Bmatrix} \beta_{G_m}(v) + p_n, & v \in V(G_m) \\ \beta_{G_n}(v) + p_m, & v \in V(G_n) \end{Bmatrix}$$

Theorem 2.4:

The Multiplicative Y-index of $G_m + G_n$ satisfies the below inequality,

$$\prod\nolimits_Y(G_m + G_n) \leq \left[\frac{Y(G_m)+4F(G_m)p_n+6M_1(G_m)p_n^2+8p_n^3 q_m+p_n^4 p_m}{p_m}\right]^{p_m} \times$$

$$\left[\frac{Y(G_n)+4F(G_n)p_m+6M_1(G_n)p_m^2+8p_m^3 q_n+p_m^4 p_n}{p_n}\right]^{p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

Proof:

Utilizing the multiplicative Y-index definition,

$$\prod_Y(G_m+G_n)=\prod_{v\in V(G_m+G_n)}\beta_{G_m+G_n}(v)^4$$

$$=\prod_{v\in V(G_m)}\left(\beta_{G_m}(v)+p_n\right)^4\prod_{v\in V(G_n)}\left(\beta_{G_n}(v)+p_m\right)^4$$

We now have, according to lemma 2.1,

$$\prod_Y(G_m+G_n)\le\left[\frac{\sum_{v\in V(G_m)}\left(\beta_{G_m}(v)+p_n\right)^4}{p_m}\right]^{p_m}\times\left[\frac{\sum_{v\in V(G_n)}\left(\beta_{G_n}(v)+p_m\right)^4}{p_n}\right]^{p_n}$$

We get the inequality. The inequality exists, according to lemma 2.1, if and only if for each $u_mv_m\in V(G_m)$ and $u_nv_n\in V(G_n)$,

$$\left(\beta_{G_m}(u_m)+p_n\right)^4=\left(\beta_{G_m}(v_m)+p_n\right)^4 \text{ and } \left(\beta_{G_n}(u_n)+p_m\right)^4=\left(\beta_{G_n}(v_n)+p_m\right)^4$$

As a result, for each $u_mv_m\in V(G_m)$ and $u_nv_n\in V(G_n)$,

$$\beta_{G_m}(u_m)=\beta_{G_m}(v_m),\beta_{G_n}(u_n)=\beta_{G_n}(v_n)$$

G_m and G_n are thus both regular graphs and we receive the complete result.

Theorem 2.5:

The Multiplicative S-index of G_m+G_n satisfies the below inequality,

$$\prod_S(G_m+G_n)\le\left[\frac{S(G_m)+5Y(G_m)p_n+10F(G_m)p_n^2+10M_1(G_m)p_n^3+10p_n^4q_m+p_n^5p_m}{p_m}\right]^{p_m}\times$$

$$\left[\frac{S(G_n)+5Y(G_n)p_m+10F(G_n)p_m^2+10M_1(G_n)p_m^3+10p_m^4q_n+p_m^5p_n}{p_n}\right]^{p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

The Cartesian product of graph

The Cartesian product $G_m\times G_n$ of graphs G_m and G_nv has the vertex set $V(G_m\times G_n)=V(G_m)\times V(G_n)$ and $(u,x)(v,y)$ is an edge of $G_m\times G_n$ if $uv\in E(G_m)$ and $x=y$, or $u=v$ and $xy\in E(G_n)$.Obviously,$|V(G_m\times G_n)|=|V(G_m)||V(G_n)|=p_mp_n$, $|E(G_m\times G_n)|=|E(G_m)||V(G_n)|+|E(G_n)||V(G_m)|=q_mp_n+q_np_m$.

$$\beta_{G_m\times G_n}(x_1,x_2)=\beta_{G_m}(x_1)+\beta_{G_n}(x_2)$$

Theorem 2.6:

The Multiplicative Y-index of $G_m\times G_n$ satisfies the below inequality,

$$\prod_Y(G_m\times G_n)\le\left[\frac{p_nY(G_m)+p_mY(G_n)+8F(G_m)q_n+6M_1(G_m)M_1(G_n)+8F(G_n)q_m}{p_mp_n}\right]^{p_mp_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

Proof:

Utilizing the multiplicative Y-index definition,

$$\prod_Y(G_m \times G_n) = \prod_{(v_m,v_n)\in V(G_m\times G_n)} \beta_{G_m\times G_n}(v_m, v_n)^4$$

$$= \prod_{v_m\in V(G_m)} \prod_{v_n\in V(G_n)} \left(\beta_{G_m}(v_m) + \beta_{G_n}(v_n)\right)^4$$

We now have, according to lemma 2.1,

$$\prod_Y(G_m \times G_n) \le \left[\frac{\sum_{v_m\in V(G_m)} \sum_{v_n\in V(G_n)} \left(\beta_{G_m}(v_m)+\beta_{G_n}(v_n)\right)^4}{p_m p_n}\right]^{p_m p_n}$$

We get the inequality. The inequality exists, according to lemma 2.1, if and only if for each $(u_m, u_n), (v_m, v_n) \in V(G)$

$$\left(\beta_{G_m}(u_m) + \beta_{G_n}(u_n)\right)^4 = \left(\beta_{G_m}(v_m) + \beta_{G_n}(v_n)\right)^4$$

As a result, for each $u_m v_m \in V(G_m)$ and $u_n v_n \in V(G_n)$,

$$\beta_{G_m}(u_m) = \beta_{G_m}(v_m), \beta_{G_n}(u_n) = \beta_{G_n}(v_n)$$

G_m and G_n are thus both regular graphs and we receive the complete result.

Theorem 2.7:

The Multiplicative S-index of $G_m \times G_n$ satisfies the below inequality,

$$\prod_S(G_m \times G_n) \le \left[\frac{p_n S(G_m) + p_m S(G_n) + 10 q_n Y(G_m) + 10F(G_m)M_1(G_n) + 10F(G_n)M_1(G_m) + 10Y(G_n)q_m}{p_m p_n}\right]^{p_m p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

The Composition of graph

The Composition $G_m[G_n]$ of graphs G_m and G_n with disjoint vertex sets $V(G_m)$ and $V(G_n)$ and edge sets $E(G_m)$ and $E(G_n)$ is the graph with vertex set $V(G_m) \times V(G_n)$ and $u = (u_1, v_1)$ is adjacent to $v = (u_2, v_2)$ whenever u_1 is adjacent to u_2 or $u_1 = u_2$ and v_1 is adjacent to v_2. $|V(G_m[G_n])| = |V(G_m)||V(G_n)| = p_m p_n$, $|E(G_m[G_n])| = |E(G_m)||V(G_n)|^2 + |V(G_m)||E(G_n)| = q_m {p_n}^2 + q_n p_m$.

$$\beta_{G_m[G_n]}(x_1, x_2) = p_n \beta_{G_m}(x_1) + \beta_{G_n}(x_2)$$

Theorem 2.8:

The Multiplicative Y-index of $G_m[G_n]$ satisfies the below inequality,

$$\prod_Y(G_m[G_n]\,) \le \left[\frac{p_n^5 Y(G_m) + p_m Y(G_n) + 8F(G_m)p_n^3 q_n + 6p_n^2 M_1(G_m)M_1(G_n) + 8F(G_n)p_n q_m}{p_m p_n}\right]^{p_m p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

Proof:

Utilizing the multiplicative Y-index definition,

$$\prod_Y(G_m[G_n]) = \prod_{(v_m,v_n)\in V(G_m[G_n])} \beta_{G_m[G_n]}(v_m, v_n)^4$$

$$= \prod_{v_m\in V(G_m)} \prod_{v_n\in V(G_n)} \left(p_n\beta_{G_m}(v_m) + \beta_{G_n}(v_n)\right)^4$$

We now have, according to lemma 2.1,

$$\prod_Y(G_m[G_n]) \leq \left[\frac{\sum_{v_m \in V(G_m)} \sum_{v_n \in V(G_n)} \left(p_n \beta_{G_m}(v_m) + \beta_{G_n}(v_n)\right)^4}{p_m p_n}\right]^{p_m p_n}$$

We get the inequality. The inequality exists, according to lemma 2.1, if and only if for each $(u_m, u_n), (v_m, v_n) \in V(G)$

$$\left(p_n \beta_{G_m}(u_m) + \beta_{G_n}(u_n)\right)^4 = \left(p_n \beta_{G_m}(v_m) + \beta_{G_n}(v_n)\right)^4$$

As a result, for each $u_m v_m \in V(G_m)$ and $u_n v_n \in V(G_n)$,

$$\beta_{G_m}(u_m) = \beta_{G_m}(v_m), \beta_{G_n}(u_n) = \beta_{G_n}(v_n)$$

G_m and G_n are thus both regular graphs and we receive the complete result.

Theorem 2.9:

The Multiplicative S-index of $G_m[G_n]$ satisfies the below inequality,

$$\prod_S(G_m[G_n]) \leq \left[\frac{p_n^6 S(G_m) + p_n^5 Y(G_m) + p_m S(G_n) + 10 p_n^2 F(G_n) M_1(G_m) + 10 p_n^3 F(G_m) M_1(G_n) + 10 Y(G_n) p_n q_m + 10 Y(G_m) p_n^4 q_n}{p_m p_n}\right]^{p_m p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

The Tensor product of graph

The Tensor product $G_m \otimes G_n$ of graphs G_m and G_n has the vertex set $V(G_m \otimes G_n) = V(G_m) \times V(G_n)$ and $(u, x)(v, y)$ is an edge of $G_m \oplus G_n$ if $uv \in E(G_m)$ and $xy \in E(G_n)$. Obviously, $|V(G_m \otimes G_n)| = |V(G_m)||V(G_n)| = p_m p_n$, $|E(G_m \otimes G_n)| = 2|E(G_m)||E(G_n)| = 2 q_m q_n$.

$$\beta_{G_m \otimes G_n}(x_1, x_2) = \beta_{G_m}(x_1) \beta_{G_n}(x_2)$$

Theorem 2.10:

The Multiplicative Y-index of $G_m \otimes G_n$ is determined by

$$\prod_Y(G_m \otimes G_n) = \prod_Y(G_m) \prod_Y(G_n)$$

Proof:

Utilizing the multiplicative Y-index definition,

$$\prod_Y(G_m \otimes G_n) = \prod_{(u_m, v_n) \in V(G_m \otimes G_n)} \beta_{G_m \otimes G_n}(v)^4$$

$$= \prod_{u_m \in V(G_m)} \left(\beta_{G_m}(u_m)\right)^4 \prod_{v_n \in V(G_n)} \left(\beta_{G_n}(u_n)\right)^4$$

We receive the complete result.

Theorem 2.11:

The Multiplicative S-index of $G_m \otimes G_n$ is determined by

$$\prod_S(G_m \otimes G_n) = \prod_S(G_m) \prod_S(G_n)$$

The Strong product of graph

The Strong product $G_m * \mathrm{G_n}$ of a graphs G_m and $\mathrm{G_n}$ is a graph with vertex set $V(G_m) \times V(\mathrm{G_n})$ and any two vertices (u_p, v_r) and (u_q, v_s) are adjacent if and only if $[u_p = u_q \ and \ v_r v_s \in E(\mathrm{G_n})]$ or $[v_r = v_s \ and \ u_p u_q \in E(G_m)]$ or $[u_p u_q \in E(G_m) \ and \ v_r v_s \in E(\mathrm{G_n})]$. $|V(G_m * \mathrm{G_n})| = |V(G_m)||V(\mathrm{G_n})| = p_m p_n$, $|E(G_m * \mathrm{G_n})| = |E(G_m)||V(\mathrm{G_n})| + |V(G_m)||E(\mathrm{G_n})| + 2|E(G_m)||E(\mathrm{G_n})| = q_m p_n + p_m q_n + 2q_m q_n$.

$$\beta_{G_m * \mathrm{G_n}}(x_1, x_2) = \beta_{G_m}(x_1) + \beta_{\mathrm{G_n}}(x_2) + \beta_{G_m}(x_1)\beta_{\mathrm{G_n}}(x_2)$$

Theorem 2.12:

The Multiplicative Y-index of $G_m * G_n$ satisfies the below inequality,

$$\prod_Y(G_m * G_n) \leq \left[\frac{\begin{array}{c} Y(G_m)[4F(G_n)+6M_1(G_n)+8q_n+p_n]+Y(n)[4F(G_m)+6M_1(G_m)+8q_m+p_m] \\ +4F(G_m)[3M_1(G_n)+2q_n]+4F(G_n)[3M_1(G_m)+2q_m]+Y(G_m)Y(G_n) \\ 6M_1(G_m)M_1(G_n)+12F(G_m)F(G_n) \end{array}}{p_m p_n}\right]^{p_m p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

Proof:

Utilizing the multiplicative Y-index definition,

$$\prod_Y(G_m * G_n) = \prod_{(v_m, v_n) \in V(G_m * G_n)} \beta_{G_m * G_n}(v_m, v_n)^4$$

$$= \prod_{v_m \in V(G_m)} \prod_{v_n \in V(G_n)} \left(\beta_{G_m}(v_m) + \beta_{G_n}(v_n) + \beta_{G_m}(v_m)\beta_{G_n}(v_n)\right)^4$$

We now have, according to lemma 2.1,

$$\prod_Y(G_m * G_n) \leq \left[\frac{\sum_{v_m \in V(G_m)} \sum_{v_n \in V(G_n)} \left(\beta_{G_m}(v_m) + \beta_{G_n}(v_n) + \beta_{G_m}(v_m)\beta_{G_n}(v_n)\right)^4}{p_m p_n}\right]^{p_m p_n}$$

We get the inequality. The inequality exists, according to lemma 2.1, if and only if for each $(u_m, u_n), (v_m, v_n) \in V(G)$

$$\left(\beta_{G_m}(u_m) + \beta_{G_n}(u_n) + \beta_{G_m}(u_m)\beta_{G_n}(u_n)\right)^4 = \left(\beta_{G_m}(v_m) + \beta_{G_n}(v_n) + \beta_{G_m}(v_m)\beta_{G_n}(v_n)\right)^4$$

As a result, for each $u_m v_m \in V(G_m)$ and $u_n v_n \in V(G_n)$,

$$\beta_{G_m}(u_m) = \beta_{G_m}(v_m), \beta_{G_n}(u_n) = \beta_{G_n}(v_n)$$

G_m and G_n are thus both regular graphs and we receive the complete result.

Theorem 2.13:

The Multiplicative S-index of $G_m * G_n$ satisfies the below inequality,

$$\prod_S(G_m * G_n) \leq \left[\frac{\begin{array}{c} S(G_m)[p_n+S(G_n)+10q_n+5Y(G_n)+10M_1(G_n)+10F(G_n)]+ \\ S(n)[p_m+10q_m+5Y(G_m)+10M_1(G_m)+10F(G_m)]+Y(G_m) \\ [10q_n+20Y(G_n)+20M_1(G_n)+30F(G_n)]+10F(G_n)M_1(G_m) \\ Y(G_m)[10q_m+20M_1(G_m)+30F(G_m)]+F(G_m)[10M_1(G_n)+30F(G_n)] \end{array}}{p_m p_n}\right]^{p_m p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

The Disjunction of graph

The Disjunction $G_m \vee \mathrm{G_n}$ of a graphs G_m and $\mathrm{G_n}$ is the graph with vertex set $V(G_m) \times V(\mathrm{G_n})$ and u_1v_1 is adjacent with u_2v_2 whenever $u_1u_2 \in E(G_m)$ and $v_1v_2 \in E(\mathrm{G_n})$. $|V(G_m \vee \mathrm{G_n})| = |V(G_m)||V(\mathrm{G_n})| = p_m p_n$, $|E(G_m \vee \mathrm{G_n})| = |E(G_m)||V(\mathrm{G_n})|^2 + |V(G_m)|^2|E(\mathrm{G_n})| - 2|E(G_m)||E(\mathrm{G_n})| = q_m p_n^2 + p_m^2 q_n - 2q_m q_n$.

$$\beta_{G_m \vee \mathrm{G_n}}(x_1, x_2) = p_n \beta_{G_m}(x_1) + p_m \beta_{\mathrm{G_n}}(x_2) - \beta_{G_m}(x_1)\beta_{\mathrm{G_n}}(x_2)$$

Theorem 2.14:

The Multiplicative Y-index of $G_m \vee G_n$ satisfies the below inequality,

$$\prod_Y(G_m \vee G_n) \le \left[\frac{\begin{matrix} p_m Y(G_n)[p_m^4 + 6p_m M_1(G_m) - 4F(G_m) - 8p_m^2 q_m] + 12 p_m p_n F(G_m) F(G_n) + \\ 4p_m^2 p_n F(G_n)[2p_m q_m - 3M_1(G_m)] + 6p_m^2 p_n^2 M_1(G_m) M_1(n) + Y(G_m) Y(G_n) + \\ p_n Y(G_m)[p_n^4 + 6p_n M_1(G_n) - 4F(G_n) - 8p_n^2 q_n] + 4p_n^2 p_m F(G_m) \\ [2p_n q_n - 3M_1(G_n)] \end{matrix}}{p_m p_n}\right]^{p_m p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

Proof:

Utilizing the multiplicative Y-index definition,

$$\prod_Y(G_m \vee G_n) = \prod_{(v_m, v_n) \in V(G_m \vee G_n)} \beta_{G_m \vee G_n}(v_m, v_n)^4$$

$$= \prod_{v_m \in V(G_m)} \prod_{v_n \in V(G_n)} \left(p_n \beta_{G_m}(v_m) + p_m \beta_{G_n}(v_n) - \beta_{G_m}(v_m)\beta_{G_n}(v_n)\right)^4$$

We now have, according to lemma 2.1,

$$\prod_Y(G_m \vee G_n) \le \left[\frac{\sum_{v_m \in V(G_m)} \sum_{v_n \in V(G_n)} \left(p_n \beta_{G_m}(v_m) + p_m \beta_{G_n}(v_n) - \beta_{G_m}(v_m)\beta_{G_n}(v_n)\right)^4}{p_m p_n}\right]^{p_m p_n}$$

We get the inequality. The inequality exists, according to lemma 2.1, if and only if for each $(u_m, u_n), (v_m, v_n) \in V(G)$

$$\left(p_n \beta_{G_m}(u_m) + p_m \beta_{G_n}(u_n) - \beta_{G_m}(u_m)\beta_{G_n}(u_n)\right)^4 = \left(p_n \beta_{G_m}(v_m) + p_m \beta_{G_n}(v_n) - \beta_{G_m}(v_m)\beta_{G_n}(v_n)\right)^4$$

As a result, for each $u_m v_m \in V(G_m)$ and $u_n v_n \in V(G_n)$,

$$\beta_{G_m}(u_m) = \beta_{G_m}(v_m), \beta_{G_n}(u_n) = \beta_{G_n}(v_n)$$

G_m and G_n are thus both regular graphs and we receive the complete result.

Theorem 2.15:

The Multiplicative S-index of $G_m \vee G_n$ satisfies the below inequality,

$$\prod_S(G_m \vee G_n) \le \left[\frac{\begin{matrix} p_n S(G_m)[p_n^4 + 10p_n^2 M_1(G_n) - 10p_n^3 q_n - 10p_n F(G_n) + 5Y(G_n)] - S(G_m) S(G_n) \\ p_m S(G_n)[p_m^4 + 10p_m^2 M_1(G_m) - 10p_m^3 q_m - 10p_m F(G_m) + 5Y(G_m)] + \\ p_m p_n Y(G_m)[10p_n^3 q_n - 20p_n^2 M_1(G_n) - 20Y(G_n) + 30p_n F(G_n)] + p_m p_n F(G_m) \\ [10p_n^2 p_m M_1(G_n) + 30p_m Y(G_n) - 30p_m p_n F(G_n)] + 10p_n^2 p_m^3 F(G_n) M_1(G_m) + \\ p_m p_n Y(G_n)[10p_m^3 q_m - 20p_m^2 M_1(G_m)] \end{matrix}}{p_m p_n}\right]^{p_m p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

The Symmetric difference of graph

The Symmetric Difference $G_m \oplus \mathrm{G_n}$ of two graphs G_m and $\mathrm{G_n}$ is a graph with vertex set $V(G_m) \times V(\mathrm{G_n})$ and $E(G_m \oplus \mathrm{G_n}) = \{(u_1,u_2)(v_1,v_2)/u_1v_1 \in E(G_m)\ or\ u_2v_2 \in E(\mathrm{G_n})\ but\ not\ both\}$ $|V(G_m \oplus \mathrm{G_n})| = |V(G_m)||V(\mathrm{G_n})| = p_m p_n$, $|E(G_m \oplus \mathrm{G_n})| = |E(G_m)||V(\mathrm{G_n})|^2 + |V(G_m)|^2|E(\mathrm{G_n})| - 4|E(G_m)||E(\mathrm{G_n})| = q_m p_n^2 + p_m^2 q_n - 4q_m q_n$.

$$\beta_{G_m \oplus \mathrm{G_n}}(x_1,x_2) = p_n\beta_{G_m}(x_1) + p_m\beta_{\mathrm{G_n}}(x_2) - 2\beta_{G_m}(x_1)\beta_{\mathrm{G_n}}(x_2)$$

Theorem 2.16:

The Multiplicative Y-index of $G_m \oplus G_n$ satisfies the below inequality,

$$\prod_Y(G_m \oplus G_n) \leq \left[\frac{\begin{array}{c} p_mY(G_n)[p_m^4+24p_mM_1(G_m)-32F(G_m)-16p_m^2q_m]+48p_mp_nF(G_m)F(G_n) \\ +8p_m^2p_nF(G_n)[2p_mq_m-3M_1(G_m)]+6p_m^2p_n^2M_1(G_m)M_1(n) \\ +p_nY(G_m)[p_n^4+24p_nM_1(G_n)-32F(G_n)-16p_n^2q_n]+8p_n^2p_mF(G_m) \\ [2p_nq_n-3M_1(G_n)]+16Y(G_m)Y(G_n) \end{array}}{p_mp_n}\right]^{p_mp_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

Proof:

Utilizing the multiplicative Y-index definition,

$$\prod_Y(G_m \oplus G_n) = \prod_{(v_m,v_n)\in V(G_m \oplus G_n)} \beta_{G_m \oplus G_n}(v_m,v_n)^4$$

$$= \prod_{v_m\in V(G_m)} \prod_{v_n \in V(G_n)} \left(p_n\beta_{G_m}(v_m) + p_m\beta_{G_n}(v_n) - 2\beta_{G_m}(v_m)\beta_{G_n}(v_n)\right)^4$$

We now have, according to lemma 2.1,

$$\prod_Y(G_m \oplus G_n) \leq \left[\frac{\sum_{v_m\in V(G_m)}\sum_{v_n\in V(G_n)}\left(p_n\beta_{G_m}(v_m)+p_m\beta_{G_n}(v_n)-2\beta_{G_m}(v_m)\beta_{G_n}(v_n)\right)^4}{p_mp_n}\right]^{p_mp_n}$$

We get the inequality. The inequality exists, according to lemma 2.1, if and only if for each $(u_m,u_n),(v_m,v_n) \in V(G)$

$$\left(p_n\beta_{G_m}(u_m) + p_m\beta_{G_n}(u_n) - 2\beta_{G_m}(u_m)\beta_{G_n}(u_n)\right)^4 = \left(p_n\beta_{G_m}(v_m) + p_m\beta_{G_n}(v_n) - 2\beta_{G_m}(v_m)\beta_{G_n}(v_n)\right)^4$$

As a result, for each $u_mv_m \in V(G_m)$ and $u_nv_n \in V(G_n)$,

$$\beta_{G_m}(u_m) = \beta_{G_m}(v_m), \beta_{G_n}(u_n) = \beta_{G_n}(v_n)$$

G_m and G_n are thus both regular graphs and we receive the complete result.

Theorem 2.17:

The Multiplicative S-index of $G_m \oplus G_n$ satisfies the below inequality,

$$\prod_S(G_m \oplus G_n) \leq \left[\frac{\begin{gathered} p_nS(G_m)[p_n^4+40p_n^2M_1(G_n)-20p_n^3q_n-80p_nF(G_n)+80Y(G_n)]+p_mS(G_n) \\ [p_m^4+40p_m^2M_1(G_m)-20p_m^3q_m-80p_mF(G_m)+80Y(G_m)]+p_mp_nY(G_m) \\ [10p_n^3q_n-40p_n^2M_1(G_n)-160Y(G_n)+120p_nF(G_n)]+p_mp_nF(G_m) \\ [10p_n^2p_mM_1(G_n)+120p_mY(G_n)-60p_mp_nF(G_n)]+10p_n^2p_m^3F(G_n)M_1(G_m)+ \\ p_mp_nY(G_n)[10p_m^3q_m-40p_m^2M_1(G_m)]-32S(G_m)S(G_n) \end{gathered}}{p_mp_n}\right]^{p_mp_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

The Corona product of graph

The Corona product $G_m \odot \mathrm{G_n}$ of graphs G_m and $\mathrm{G_n}$ with disjoint vertex sets $V(G_m)$ and $V(\mathrm{G_n})$ and edge sets $E(G_m)$ and $E(\mathrm{G_n})$ is the graph obtained by one copy of G_m and k_1 copies of $\mathrm{G_n}$ and joining the i^{th} vertex of G_m to every vertex in i^{th} copy of $\mathrm{G_n}$. Obviously, $|V(G_m \odot \mathrm{G_n})| = |V(G_m)| + |V(G_m)||V(\mathrm{G_n})| = p_m + p_mp_n$, $|E(G_m \odot \mathrm{G_n})| = |E(G_m)| + |V(G_m)||E(\mathrm{G_n})| + |V(G_m)||V(\mathrm{G_n})| = q_m + p_mq_n + p_mp_n$.

$$\beta_{G_m \odot \mathrm{G_n}}(v) = \begin{Bmatrix} \beta_{G_m}(v) + p_n, & v \in V(G_m) \\ \beta_{\mathrm{G_n}}(v) + 1, & v \in V(\mathrm{G_n}) \end{Bmatrix}$$

Theorem 2.18:

The Multiplicative Y-index of $G_m \odot G_n$ satisfies the below inequality,

$$\prod_Y(G_m \odot G_n) \leq \left[\frac{Y(G_m)+4F(G_m)p_n+6M_1(G_m)p_n^2+8p_n^3q_m+p_n^4p_m}{p_m}\right]^{p_m} \times$$

$$\left[\frac{Y(G_n)+4F(G_n)+6M_1(G_n)+8q_n+p_n}{p_n}\right]^{p_mp_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

Proof:

Utilizing the multiplicative Y-index definition,

$$\prod_Y(G_m \odot G_n) = \prod_{v \in V(G_m \odot G_n)} \beta_{G_m \odot G_n}(v)^4$$

$$= \prod_{v \in V(G_m)} \left(\beta_{G_m}(v) + p_n\right)^4 \times \left[\prod_{v \in V(G_n)} \left(\beta_{G_n}(v) + 1\right)^4\right]^{p_m}$$

We now have, according to lemma 2.1,

$$\prod_Y(G_m \odot G_n) \leq \left[\frac{\sum_{v \in V(G_m)} \left(\beta_{G_m}(v)+p_n\right)^4}{p_m}\right]^{p_m} \times \left[\frac{\sum_{v \in V(G_n)} \left(\beta_{G_n}(v)+1\right)^4}{p_n}\right]^{p_mp_n}$$

We get the inequality. The inequality exists, according to lemma 2.1, if and only if for each $u_mv_m \in V(G_m)$ and $u_nv_n \in V(G_n)$,

$$\left(\beta_{G_m}(u_m) + p_n\right)^4 = \left(\beta_{G_m}(v_m) + p_n\right)^4 \text{ and } \left(\beta_{G_n}(u_n) + 1\right)^4 = \left(\beta_{G_n}(v_n) + 1\right)^4$$

As a result, for each $u_mv_m \in V(G_m)$ and $u_nv_n \in V(G_n)$,

$$\beta_{G_m}(u_m) = \beta_{G_m}(v_m), \beta_{G_n}(u_n) = \beta_{G_n}(v_n)$$

G_m and G_n are thus both regular graphs and we receive the complete result.

Theorem 2.19:

The Multiplicative S-index of $G_m \odot G_n$ satisfies the below inequality,

$$\prod_S(G_m \odot G_n) \leq \left[\frac{S(G_m)+5Y(G_m)p_n+10F(G_m)p_n^2+10M_1(G_m)p_n^3+10p_n^4q_m+p_n^5p_m}{p_m}\right]^{p_m} \times$$

$$\left[\frac{S(G_n)+5Y(G_n)+10F(G_n)+10M_1(G_n)+10q_n+p_n}{p_n}\right]^{p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

The Corona join product of graph

Let $G_m(k_1, j_1)$ and $G_n(k_2, j_2)$ be simple connected graphs, and the Corona join graph of G_m and G_n is obtained by taking one copy of G_m, k_1 copies of G_n, and joining each vertex of the i^{th} copy of G_n with all vertices of G_m. The degree of a vertex v of $G_m \oplus G_n$ is defined as:

$$\beta_{G_m \oplus \mathrm{G_n}}(v) = \begin{Bmatrix} \beta_{G_m}(v) + p_m p_n\,, & v \in V(G_m) \\ \beta_{\mathrm{G_n}}(v) + p_m\,, & v \in V(\mathrm{G_n}) \end{Bmatrix}$$

Theorem 2.20:

The Multiplicative Y-index of $G_m \oplus G_n$ satisfies the below inequality,

$$\prod_Y(G_m \oplus G_n) \leq \left[\frac{Y(G_m)+4F(G_m)p_m p_n+6M_1(G_m)p_m^2p_n^2+8p_m^3p_n^3q_m+p_n^4p_m^5}{p_m}\right]^{p_m} \times$$

$$\left[\frac{Y(G_n)+4F(G_n)p_m+6M_1(G_n)p_m^2+8p_m^3q_n+p_m^4p_n}{p_n}\right]^{p_m p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

Proof:

Utilizing the multiplicative Y-index definition,

$$\prod_Y(G_m \oplus G_n) = \prod_{v \in V(G_m \oplus G_n)} \beta_{G_m \oplus G_n}(v)^4$$

$$= \prod_{v \in V(G_m)} \left(\beta_{G_m}(v) + p_m p_n\right)^4 \times \left[\prod_{v \in V(G_n)} \left(\beta_{G_n}(v) + p_m\right)^4\right]^{p_m}$$

We now have, according to lemma 2.1,

$$\prod_Y(G_m \oplus G_n) \leq \left[\frac{\sum_{v \in V(G_m)} \left(\beta_{G_m}(v) + p_m p_n\right)^4}{p_m}\right]^{p_m} \times \left[\frac{\sum_{v \in V(G_n)} \left(\beta_{G_n}(v) + p_m\right)^4}{p_n}\right]^{p_m p_n}$$

We get the inequality. The inequality exists, according to lemma 2.1, if and only if for each $u_m v_m \in V(G_m)$ and $u_n v_n \in V(G_n)$,

$$\left(\beta_{G_m}(u_m) + p_m p_n\right)^4 = \left(\beta_{G_m}(v_m) + p_m p_n\right)^4 \text{ and } \left(\beta_{G_n}(u_n) + p_m\right)^4 = \left(\beta_{G_n}(v_n) + p_m\right)^4$$

As a result, for each $u_m v_m \in V(G_m)$ and $u_n v_n \in V(G_n)$,

$$\beta_{G_m}(u_m) = \beta_{G_m}(v_m), \beta_{G_n}(u_n) = \beta_{G_n}(v_n)$$

G_m and G_n are thus both regular graphs and we receive the complete result.

Theorem 2.21:

The Multiplicative S-index of $G_m \oplus G_n$ satisfies the below inequality,

$$\prod_S(G_m \oplus G_n) \leq \left[\frac{S(G_m)+5Y(G_m)p_m p_n+10F(G_m)p_m^2 p_n^2+10M_1(G_m)p_m^3 p_n^3+10p_m^4 p_n^4 q_m+p_n^5 p_m^6}{p_m}\right]^{p_m} \times$$

$$\left[\frac{S(G_n)+5Y(G_n)p_m+10F(G_n)p_m^2+10M_1(G_n)p_m^3+10p_m^4 q_n+p_m^5 p_n}{p_n}\right]^{p_n}$$

The equality holds if and only if G_m and G_n are regular graphs.

2. Conclusion

Topological indices are defined and used in many fields to investigate the properties of various objects such as atoms and molecules. Mathematicians and chemists have defined and studied a number of topological indices. We investigated upper bound for the Multiplicative Y -index and Multiplicative S-index of various graph operations such as Join, Cartesian product, Composition, Tensor product, Strong product, Disjunction, Symmetric difference, Corona product, Corona join product and few well known graphs are evaluated in this work.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

References

[1] A. Alameri, N. AI-Naggar, M. AI-Rumaima, M. Alsharafi, Y-index of some graph operations, Int. J. Appl. Eng. Res. 15 (2) (2020) 173-179.

[2] Y. Asghar, Ali Iranmanesh, A Multiplicative version of forgotten topological index, Math. Interdis. Res. 4 (2019) 193-211.

[3] N. De, S. Nayeem, A. Pal, The F-index of some graph operations, Discrete Math. Algo. Appl. 8 (2) (2016) 1650025.

[4] B. Furtula, I. Gutman, A forgotten topological index, J. Math. Chem. 53 (4) (2015) 1184-1190.

[5] I. Gutman, N. Trinajstic, Graph theory and molecular orbitals. Total ϕ-electron energy of alternant hydrocarbons. Chemical Physics Letters 17 (4) (1972) 535–538.

[6] A. Ilic and B. Zhou, On reformulated Zagreb indices, Discrete. Appl. Math. 160 (2012) 204-209.

[7] C. D. Kinkar, Y. Aysun, T. Muge, The multiplicative Zagreb indices of graph operations, Jour. Inequ. and Appl. (2013) 1-14.

[8] S. Nagarajan, G. Kayalvizhi, G. Priyadharsini, S-index of different graph operations, Asian. Res. Jour. of Math. 17 (12) (2021) 43-52.

[9] De. Nilanjan, Sk. Md. Abu Nayeem, Reformulated first Zagreb index of some graph operations, Mathematics. 3 (2015) 945-960.

[10] G. Su, L. Xiong, On the maximum and minimum first reformulated Zagreb index with connectivity atmost K, Filomat. 2 (2011) 75-83.

[11] R. Todeschini, V. Consonni, New local vertex invarients and molecular descriptors based on functions of the vertex degrees, MATCH Commun. Math. Comput. Chem. 64 (2010) 359-372.

[12] K. Xu, K. C. Das, K. Tang, On the multiplicative Zagreb coindex of graphs, Opuscula Math. 33 (1) (2013) 191-204.

[13] B. Zhou, N. Trinajstic, Some properties of the reformulated Zagreb indices, J. Math. Chem. 48 (2010) 714-719.