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# Identities of matrix powers of a matrix and matrix power of matrix on lie groups

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## Abstract

This paper introduces fundamental definitions pertaining to Matrix Lie Groups and Lie Algebra. We illustrate the concepts of the exponential of a matrix and the logarithm of a matrix as integral components of our discourse. We introduce novel identities related to the matrix powers of a matrix. To prove these identities, we employ the results of the matrix powers of a matrix. Finally, we derive an expression for the matrix powers of a matrix within the context of a connected matrix lie group.

**Keywords:** Matrix exponential; Matrix logarithm; Matrix powers of matrix; Matrix Lie group

## 1. Introduction

This paper is related to matrix functions, which play a vital role in science and engineering, yielding increasingly fascinating results. In various scientific and engineering fields, matrix functions are widely used. They are especially important in numerical methods and solving systems of differential equations. In general, series are commonly utilized in the construction of numerical algorithms. The fundamental matrix functions, namely, matrix exponential [1] and matrix logarithm [2] were typically defined through the Taylor series expansion of a function. Significant advancements pertaining to the matrix exponentials and matrix logarithms have been made over the past few decades. The matrix powers of a matrix [3] was recently defined by using the properties of functions of matrix exponentials and matrix logarithms. Metrices are extensively applied in other subject areas in mathematics such as group theory. Hence, the matrix powers of a matrix could be applied to many subject areas in mathematics. Some explicit identities related to the matrix powers of a matrix have already been introduced. In this paper, we deduce several novel identities related to matrix powers of a matrix. Lastly, we explore connections between the matrix powers of a matrix and on a connected matrix Lie group.

## 2. Basic definitions related to Matrix Lie Groups and Lie Algebra [4]

The following definitions that are related to the Matrix Lie Groups and Lie Algebra are important obtaining expression for matrix power of matrix on connected matrix Lie groups.

**Definition 1.** A group is a set $G$ together with a map $*: \ G \times G \ \rightarrow \ G$

$[* \ (g_1, g_2) \ \longmapsto \ g_1 * g_2]$ such that

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1. $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$ for all $g_1, g_2, g_3 \in G$;
2. There exists $e \in G$ such that $e * g = g * e = g$ for each $g \in G$;
3. For each $g \in G$ there exists $h \in G$ such that $h * g = g * h = e$.

The general linear groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ are given below. The product is given by matrix multiplication, and matrix multiplication is associative.

$GL(n,\mathbb{R}) := \{A \in M_n(\mathbb{R}) : A^{-1}\ exists\}$ is the general linear group over $\mathbb{R}$.

$GL(n,\mathbb{C}) := \{A \in M_n(\mathbb{C}) : A^{-1}\ exists\}$ is the general linear group over $\mathbb{C}$.

**Definition 2**. Let $A_n$ be a sequence of complex matrices. We say that $A_n$ converges to a matrix $A$ if each entry of $A_n$ converges to the corresponding entry of $A$.

**Definition 3**. $A$ matrix Lie group is any subgroup $H$ of $GL(n,\mathbb{C})$ with the property that if $A_n$is a sequence of matrices in $H$ and $A_n$ converges to a matrix $A$, then either $A$ belongs to $H$ or $A$ is not invertible.

Essentially, what the definition of a matrix Lie group requires is that $H$ be a closed subset of $GL(n,\mathbb{C})$.

**Definition 4**. A matrix Lie Group $G$ is connected if for each $E, F \in G$, there is a continuous path $A : [a,b] \to G$, such that $A(t) \in G$ for each $t$, $A(a) = E$, and $A(b) = F$.

**Definition 5.** Let $G$ be a matrix Lie group. The Lie algebra of $G$, denoted $g$, is the set of all matrices $X$ such that $e^{tX}$ belongs to $G$ for each real number $t$.

## 3. Matrix exponential and matrix logarithm

The following definitions and properties are required for establishing our new identities.

### 3.1. Definition of Matrix exponential

By using Taylor series of exponential function, the exponential of a matrix $A$ is defined as $e^A = I + \sum_{n\geq 1} \frac{1}{n!} A^n$ [1].

### 3.2. Definition of Matrix logarithm

The logarithm of a matrix $A$ is defined by $log(A) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} (A - I)^n$, where $\|A - I\| < 1$ [2].

### 3.3. Properties of Matrix exponential and Matrix logarithm [4], [5]

Let $A, B \in M_n(\mathbb{C})$.

a) $\lim_{n\to\infty} \left(1 + \frac{A}{n}\right)^n = e^A$.
b) $\lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n = e^{A+B}$.
c) If $[A,B] \equiv AB - BA = 0$, then $e^{A+B} = e^A e^B = e^B e^A$.
d) If $\big[A,[A,B]\big] = \big[B,[A,B]\big] = 0$, then $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$.
e) $\det(e^A) = \mathrm{e}^{trace(A)}$.
f) $B\log(A) = \log(A)B$, if $AB = BA$ and $\|B\| < 1$.

### 3.4. Definition of Matrix power of a Matrix [3]

The matrix power of $A$ to $B$ is defined by $A^B = \exp(B\log(A))$.

## 4. Important result in connected matrix Lie group [4]

If $G$ is a connected matrix Lie group, then every element $A$ of $G$ can be written in the form $A = e^{X_1} e^{X_2} \dots e^{X_m}$ for some $X_1, X_2, \dots, X_m$ in Lie algebra $g$ of $G$.

## 5. Recent advancements

In our latest findings, we establish matrix power identities that are associated with the matrix exponential identities. Additionally, we derive a novel expression for matrix power of matrix on a connected Lie group.

### 5.1. Results of matrix powers of a matrix

Let $A \in N_{M_n(\mathbb{k})}(I,1)$ and $B, C \in GL(n,\mathbb{k})$ , where $\mathbb{k} = \mathbb{R}$, the real numbers, or $\mathbb{k} = \mathbb{C}$, the complex numbers, and $N_{M_n(\mathbb{k})}(I,1) = \{ A \in M_n(\mathbb{k}) : \|A - I\| < 1\}$.

- a) $\lim_{n\to\infty}\left(1+\frac{Blog(A)}{n}\right)^n = A^B$.
- b) $\lim_{n\to\infty}\left(A^{\frac{B}{n}} A^{\frac{C}{n}}\right)^n = A^{B+C}$, if $[B,C]=0$, $[A,C]=0$, $[A,B]=0$, $\|B\| < 1$ and $\|C\| < 1$.
- c) If $\left[B,[B,C]\right] = \left[C,[B,C]\right] = 0$ , $\|B\| < 1$ and $\|C\| < 1$, then $A^B A^C = A^{B+C+\frac{1}{2}[B,C]}$.
- d) $\det(A^B) = \mathrm{e}^{trace(Blog\,(A))}$.

Proof of a): $\lim_{n\to\infty}\left(1+\frac{Blog(A)}{n}\right)^n = e^{Blog(A)}$ ( by 3.3 a)).

$$= A^B \blacksquare$$

Proof of b): Supose $[B,C]=0$, $[A,C]=0$, $[A,B]=0$, $\|B\| < 1$ and $\|C\| < 1$.

Then, $[Blog(A), Clog(A)] = 0$. (By 3.3 c) and 3.3 f))

$$\lim_{n\to\infty}\left(A^{\frac{B}{n}} A^{\frac{C}{n}}\right)^n = \lim_{n\to\infty}\left(e^{\frac{Blog(A)}{n}} e^{\frac{Clog(A)}{n}}\right)^n.$$

$$= e^{Blog(A)+Clog(A)} \text{ ( by 3.3 b)).}$$

$$= e^{Blog(A)} e^{Clog(A)} \text{ (since, } [Blog(A), Clog(A)] = 0.)$$

$$= A^{B+C}. \blacksquare$$

Proof of c): Suppose $\left[B,[B,C]\right] = \left[C,[B,C]\right] = 0$ , $\|B\| < 1$ and $\|C\| < 1$.

Then, $\left[Blog(A), [Blog(A), Clog(A)]\right] = \log(A)\left[B,[B,C]\right] = 0$ and

$\left[Clog(A), [Blog(A), Clog(A)]\right] = \log(A)\left[C,[B,C]\right] = 0$. Hence,

$$A^B A^C = e^{Blog(A)} e^{Clog(A)}.$$

$$= e^{Blog(A)+Clog(A)+\frac{1}{2}[Blog(A),Clog(A)]} \text{ (By 3.3 d))}$$

$$= e^{Blog(A)+Clog(A)+\frac{log(A)}{2}[B,C]}.$$

$$= e^{(B+C+\frac{1}{2}[B,C])log(A)}.$$

$$= A^{B+C+\frac{1}{2}[B,C]}. \blacksquare$$

Proof of d): $\det(A^B) = \det\left(e^{Blog(A)}\right) = e^{trace\left(Blog(A)\right)}$(By 3.3 e)). $\blacksquare$

### 5.2. Matrix power of a matrix on a connected matrix Lie group

If $G$ is a connected matrix Lie group, then every element $A$ of $G$ and $B \in GL(n, \mathbb{k})$, $A^B$ can be written in the form $A^B = e^{B\log(e^{X_1}e^{X_2}\ldots e^{X_m})}$ for some $X_1, X_2, \ldots, X_m$ in Lie algebra $g$ of $G$. Further, if $[\sum_{i=1}^{r-1} X_i, X_r] = 0$ for $r = 2,3 \ldots m$, then $A^B = e^{BX_1}e^{BX_2} \ldots e^{BX_m}$.

Proof: Suppose $G$ is a connected matrix Lie group. Let $A \in G$ and $B \in GL(n, \mathbb{k})$. Then, $A = e^{X_1}e^{X_2} \ldots e^{X_m}$ for some $X_1, X_2, \ldots, X_m$ in Lie algebra $g$ of $G$ (By the important result in connected matrix Lie group). For any element $X$ in the connected Lie group $G, X = e^Y$ and $Y = \log(e^X)$. Hence, $A^B = e^{B\log(A)} = e^{B\log(e^{X_1}e^{X_2}\ldots e^{X_m})} = e^{BX_1}e^{BX_2} \ldots e^{BX_m}$, if $[\sum_{i=1}^{r-1} X_i, X_r] = 0$ for $r = 2,3 \ldots m$. ■

## 6. Conclusion

This study establishes identities of the matrix powers of a matrix corresponding to certain matrix exponential identities. Additionally, we derive an expression for the matrix powers of a matrix within a connected matrix Lie group.

## Compliance with ethical standards

### *Disclosure of conflict of interest*

No conflict of interest to be disclosed.

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